

# The Generalized Uncertainty Principle and Black Hole Entropy in Tunneling formalism

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## Abstract

In this Letter we study the effects of the Generalized Uncertainty Principle in the tunneling formalism for Hawking radiation to evaluate the quantum-corrected Hawking temperature and entropy for a Schwarzschild black hole. We compare our results with the existing results given by other candidate theories of quantum gravity. In the entropy-area relation we found some new corection terms and in the leading order we found a term which varies as  $\sim \sqrt{Area}$ . We also get the well known logarithmic correction in the sub-leading order. We discuss the significance of this new quantum corrected leading order term.

Keywords: black hole entropy, tunneling , generalized uncertainty principle

The realization that black holes are thermodynamic objects with well defined entropy and temperature is one of the landmark achievement in theoretical physics [1, 2]. Hawking [2] has shown that a Schwarzschild black hole has a thermal radiation with a temperature  $T_H = \frac{\hbar}{8\pi M}$ , where  $M$  is the mass of the black hole. Also the entropy associated with a Schwarzschild black hole is given by the Bekenstein-Hawking entropy-area relation

$$S = \frac{A}{4l_p^2} . \quad (1)$$

Here  $A$  is the area of the black hole horizon and  $l_p$  is the Planck length. Two leading candidate theories of quantum gravity namely, string theory and loop quantum gravity, both achieved an enormous amount of success in statistical explanation of the entropy-area law (we can see [3] for a brief overview). In this discussion we will mainly focus on the quantum-corrected

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entropy. Various theories of quantum gravity (e.g., [4, 5, 6, 7, 8]) have predicted the following expansive form for the quantum-corrected entropy-area relation:

$$S = \frac{A}{4l_p^2} + c_0 \ln \left( \frac{A}{4l_p^2} \right) + \sum_{n=1}^{\infty} c_n \left( \frac{A}{4l_p^2} \right)^{-n} + \text{const.} , \quad (2)$$

where the coefficients  $c_n$  can be regarded as model dependent parameters. Many researchers have expressed a vested interest in fixing  $c_0$  (the coefficient of the subleading logarithmic term) [4]. Recent rigorous calculations of loop quantum gravity predicts the value of  $c_0$  to be  $-1/2$  [8].

Quite recently there were some proposals to view Hawking radiation as a tunneling process in semiclassical quantum mechanics [9, 10]. The process is analogous to electron-positron pair production in a constant electric field in Schwarzschild like spacetimes. The energy of the particles changes its sign immediately after crossing the horizon neutralizing the total energy to be zero. In the formalism developed in [9, 10] the particles can follow the classically forbidden paths just after crossing the horizon. The tunneling amplitude depends on the single particle classical action which becomes complex for the outgoing particle only. The calculation involves the imaginary part of the action which is related to the Boltzmann factor for the s-wave emission process across the horizon at the Hawking temperature. Using a semiclassical method based on complex path analysis one can show particle production in Schwarzschild like spacetimes with a horizon thereby recovering the Hawking radiation. Srinivasan and Padmanabhan [10] applied the Hamilton-Jacobi method for the computation of the imaginary part of the action. Simultaneously Parikh and Wilczek [9] used the method of radial null geodesic for the same purpose which was later used by authors in [11] for calculating the Hawking temperature for different spacetimes. Recent developments although semiclassical include the same scheme for Dirac particles [12]. Authors in [13, 14] made rigorous attempts for a detailed analysis. One can include the effects of back reaction and get the quantum corrections to the temperature and entropy [15]. An extension of the analysis for non-commutative Schwarzschild spacetime was studied in [16]. A brief knowledge on the recent developments in this area can be found in [17, 18].

For the study of black hole entropy we can also use a model independent concept namely the Generalized Uncertainty Principle or GUP. The idea that the uncertainty principle could be affected by gravity was given by Mead [19]. In the regime when the gravity is strong enough, conventional Heisenberg uncertainty relation is no longer satisfactory (though approximately but perfectly valid in low gravity regimes). Later modified commutation relations between position and momenta commonly known as Generalized Uncertainty Principle were given by candidate theories of quantum gravity (String Theory, Doubly Special Relativity (or DSR) Theory and Black Hole Physics) with the prediction of a minimum measurable length [20, 21, 22]. Similar kind of commutation relation can also be found in the context of Polymer Quantization in terms of polymer mass scale [23]. Importance of the GUP can also be realized on the basis of simple *gedanken* experiments without any reference of a particular fundamental theory [21]. So we can think the GUP as a model independent concept, ideally perfect for the study of black hole entropy. Many authors have applied the GUP for a heuristic analysis of the black hole entropy (we can see [5, 24, 25, 26] for a brief idea). The authors in [27] proposed a GUP which is consistent with DSR theory, string theory and

black hole physics and which says

$$[x_i, x_j] = [p_i, p_j] = 0 \quad , \quad (3)$$

$$[x_i, p_j] = i\hbar \left[ \delta_{ij} - l \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) + l^2 (p^2 \delta_{ij} + 3p_i p_j) \right] \quad , \quad (4)$$

$$\begin{aligned} \delta x \delta p &\geq \frac{\hbar}{2} [1 - 2l\langle p \rangle + 4l^2\langle p^2 \rangle] \\ &\geq \frac{\hbar}{2} \left[ 1 + \left( \frac{l}{\sqrt{\langle p^2 \rangle}} + 4l^2 \right) (\delta p)^2 + 4l^2\langle p \rangle^2 - 2l\sqrt{\langle p^2 \rangle} \right] \quad , \end{aligned} \quad (5)$$

where  $l = \frac{l_0 l_p}{\hbar}$ . Here  $l_p$  is the Plank length ( $\approx 10^{-35}m$ ). It is normally assumed that the dimensionless parameter  $l_0$  is of the order unity. If this is the case then the  $l$  dependent terms are only important at or near the Plank regime. But here we expect the existence of a new intermediate physical length scale of the order of  $l\hbar = l_0 l_p$ . We also note that this unobserved length scale cannot exceed the electroweak length scale [27] which implies  $l_0 \leq 10^{17}$ . These equations are approximately covariant under DSR transformations but not Lorentz covariant [22]. These equations also imply

$$\delta x \geq (\delta x)_{min} \approx l_0 l_p \quad (6)$$

and

$$\delta p \leq (\delta p)_{max} \approx \frac{M_p c}{l_0} \quad (7)$$

where  $M_p$  is the Plank mass and  $c$  is the velocity of light in vacuum. With a lower bound for position fluctuations we can claim that there is a minimum measurable distance and from an upper bound of momentum fluctuations we claim that momentum measurements cannot be arbitrarily imprecise. The effect of this proposed GUP is well studied recently for some well known physical systems in [27, 28, 29].

In this Letter we study the implications of the Generalized Uncertainty Principle in the tunneling formalism to evaluate the quantum-corrected Hawking temperature and entropy for a Schwarzschild black hole. We compare our results with the existing results in the literature as we found some new corrections to the temperature and the entropy-area relation. We conclude with some comment and discussion.

We start with the formalism developed in [9] (the method with radial null geodesic) which allows one to view Hawking radiation as quantum tunneling. For our purpose let us consider a general class of spherically symmetric static spacetime with line element

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2 \quad (8)$$

with the horizon located at  $r = r_H$  such that  $f(r) = g(r) = 0$ . In the Painleve coordinates [30] there is no singularity at the morizon but the metric do not remain static as in those coordinates (8) becomes

$$ds^2 = -f(r) dt^2 + 2 f(r) \sqrt{\frac{1 - g(r)}{f(r) g(r)}} dr + dr^2 + r^2 d\Omega^2 \quad . \quad (9)$$

For the radial null geodesic we can find

$$\dot{r} = \sqrt{\frac{f(r)}{g(r)}} (\pm 1 - \sqrt{1 - g(r)}) \quad (10)$$

where  $+$ ( $-$ ) corresponds to outgoing (incoming) geodesics.  $f(r)$  and  $g(r)$  can be expanded about the horizon  $r_H$  and this helps us to approximately write  $\dot{r}$  as

$$\dot{r} \simeq \frac{1}{2} \sqrt{f'(r_H) g'(r_H)} (r - r_H) \quad (11)$$

The imaginary part of the action can be written as [9]

$$\text{Im } S = \text{Im} \int_{r_{in}}^{r_{out}} \int_{r_H}^{r_0} \frac{dH'}{\dot{r}} dr \quad (12)$$

If we equate the tunneling probability ( $\Gamma \sim e^{-\frac{2}{\hbar} \text{Im } S}$ ) with the Boltzmann factor ( $e^{-\frac{\omega}{T}}$ ) we can get the Hawking temperature as

$$T_H = \frac{\omega \hbar}{2 \text{Im } S} = \frac{\hbar}{4\pi} \sqrt{f'(r_H) g'(r_H)} \quad (13)$$

which for a Schwarzschild black hole is  $\frac{\hbar}{8\pi M}$ . There is a ‘factor of 2’ ambiguity as discussed in [31] as  $2 \text{Im } S$  is not canonically invariant but  $\text{Im } S$  is and if we take  $\text{Im } S$  in our calculation, the Hawking temperature will get amplified by a factor of 2. Now we will apply the Hamilton-Jacobi method as developed in [10] for the calculation of the imaginary part of the action and thereby calculate the Hawking temperature and entropy for a Schwarzschild black hole. The method has an advantage over the radial null geodesic method as this is valid for massive particles also and there is no ambiguity of the ‘factor 2’ discussed. All possible quantum corrections can also be incorporated with the method portrayed in [17]. For simplicity we consider a massless particle in the spacetime described by eqn. (8). The Klien-Gordon equation is given by

$$-\frac{\hbar^2}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu) \phi = 0 \quad (14)$$

We can use the ansatz

$$\phi(r, t) = e^{-\frac{i}{\hbar} S(r, t)} \quad (15)$$

for the semiclassical wave function of the Klien-Gordon equation. To incorporate the quantum corrections in powers of  $\hbar$  we can expand  $S(r, t)$  as

$$S(r, t) = S_0(r, t) + \sum_i \hbar^i S_i(r, t) \quad (16)$$

where  $S_0$  is the semiclassical value of the action. If we put this expansion in the Klien-Gordon equation we get an identical set of equations for all powers in  $\hbar$  and which says

$$\frac{\partial S_0}{\partial t} = \pm \sqrt{f(r) g(r)} \frac{\partial S_0}{\partial r} \quad \text{for} \quad \hbar^0 \quad (17)$$

$$\begin{aligned}\frac{\partial S_1}{\partial t} &= \pm \sqrt{f(r) g(r)} \frac{\partial S_1}{\partial r} && \text{for} && \hbar^1 \\ \frac{\partial S_2}{\partial t} &= \pm \sqrt{f(r) g(r)} \frac{\partial S_2}{\partial r} && \text{for} && \hbar^2 \quad \text{etc.}\end{aligned}$$

So we have some reason to conclude that  $S_i$ 's are not independent but they are proportional to  $S_0$ . Using dimensional analysis it can be shown that the most general form of eqn. (16) can be written as

$$S(r, t) = \left[ 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right] S_0(r, t) \quad (18)$$

where  $M$  is the mass of the black hole and considering the choice of unit where  $G = c = k_B = 1$  with  $\beta_i$ 's being the dimensionless constant parameters. If we solve eqn. (17) then we can find the solution for  $S(r, t)$ . The spacetime of (8) has timelike killing vectors so we would like to go for a solution

$$S_0(r, t) = \omega t + \tilde{S}_0(r) \quad (19)$$

where  $\omega$  is the energy of the particle. Now it is interesting to note that we are studying a process that occurs at the horizon and so this energy of the particle should receive some corrections due to the strength of gravity. Though the argument is heuristic but still we have some reason to believe this. Recently in [32] it is shown that the effect of the generalized uncertainty principle becomes more and more important as one approaches the event horizon. Now we see that eqns. (4) and (5) represents modified Heisenberg algebra. But the interesting part of these two relation is the term which is linear in  $l (= l_0 l_p / \hbar)$  with  $p$ . Inspired by this idea, for our purpose we will consider the generalized Heisenberg algebra (Generalized Heisenberg principle) with a small change in notation where  $x$  and  $p$  obeys the relation ( $\alpha > 0$ )

$$\delta x \delta p \geq \hbar \left[ 1 - \frac{\alpha l_p}{\hbar} \delta p + \frac{\alpha^2 l_p^2}{\hbar^2} (\delta p)^2 \right] . \quad (20)$$

In writing equation (20) we made an approximation that  $(\delta p) \approx \sqrt{\langle p^2 \rangle}$ . This means  $\langle p \rangle \approx 0$ . Now this seems to be a valid approximation as we are going to study the Schwarzschild black hole which is spherically symmetric <sup>2</sup>. We can see that if  $\alpha = 2l_0$  this is the same relation as that of (5). Here  $\delta x$  and  $\delta p$  are the position and momentum uncertainty for a quantum particle and  $\alpha$  is a dimensionless positive parameter (also known as deformation parameter in the literature of non-commutative geometry). As  $l_p = \sqrt{\frac{\hbar G}{c^3}}$ , where  $G$  is the Newtonian coupling constant, we can imply that the extra terms in the uncertainty relation is a consequence of gravity. We can re express the generalized Heisenberg principle (or GUP) of (20) in the following form

$$\delta p \geq \frac{\hbar (\delta x + \alpha l_p) - \hbar \sqrt{(\delta x + \alpha l_p)^2 - 4\alpha^2 l_p^2}}{2\alpha^2 l_p^2} , \quad (21)$$

where a negative sign choice is made by taking the classical limit. As  $l_p$  is normally viewed as an ultraviolet cut-off on spacetime geometry (e.g., [33]), it is quite justified that we can

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<sup>2</sup>Also in many problems of usual quantum mechanics we find  $\langle p \rangle = \langle x \rangle = 0$  (for ex. ground state of harmonic oscillator).

consider the dimensionless ratio  $\frac{l_p}{\delta x}$  relatively small as compared to unity. So we can Taylor expand equation (21) and rewrite the same equation after some simple manipulation as

$$\delta p \geq \frac{1}{\delta x} \left[ 1 - \frac{\alpha l_p}{2(\delta x)} + \frac{\alpha^2 l_p^2}{2(\delta x)^2} - \frac{\alpha^3 l_p^3}{2(\delta x)^3} + \frac{9}{16} \frac{\alpha^4 l_p^4}{(\delta x)^4} - \dots \right] . \quad (22)$$

The Heisenberg uncertainty principle ( $\delta p \delta x \geq 1$ ) can be translated to the lower bound  $\omega \delta x \geq 1$  with the arguments used in [34, 7], where  $\omega$  is the energy of a quantum particle. If we imply our GUP, we can rebuild the lower bound as

$$\omega_G \geq \omega \left[ 1 - \frac{\alpha l_p}{2(\delta x)} + \frac{\alpha^2 l_p^2}{2(\delta x)^2} - \frac{\alpha^3 l_p^3}{2(\delta x)^3} + \frac{9}{16} \frac{\alpha^4 l_p^4}{(\delta x)^4} - \dots \right] , \quad (23)$$

where  $\omega_G$  is the GUP corrected energy. So now with the mentioned arguments we re-write eqn. (19) as

$$S_0(r, t) = \omega_G t + \tilde{S}_0(r) . \quad (24)$$

With eqn. (17) and (24) we can write

$$\tilde{S}_0(r) = \pm \omega_G \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} . \quad (25)$$

+ (-) denotes the incoming (outgoing) of the particle. So now

$$S(r, t) = \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \left( \omega_G t \pm \omega_G \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right) . \quad (26)$$

Now we can write the solutions of the Klein-Gordon equation for the incoming and outgoing part with eqns. (15) and (26) as

$$\phi_{in} = \exp \left[ -\frac{i}{\hbar} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \left( \omega_G t + \omega_G \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right) \right] \quad (27)$$

and

$$\phi_{out} = \exp \left[ -\frac{i}{\hbar} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \left( \omega_G t - \omega_G \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right) \right] . \quad (28)$$

The incoming and outgoing probabilities of the particle can be calculated from the solutions of the Klein-Gordon equation and are given by

$$P_{in} = |\phi_{in}|^2 = \exp \left[ \frac{2}{\hbar} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \left( \omega_G \text{Im } t + \omega_G \text{Im} \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right) \right] \quad (29)$$

and

$$P_{out} = |\phi_{out}|^2 = \exp \left[ \frac{2}{\hbar} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \left( \omega_G \text{Im } t - \omega_G \text{Im} \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right) \right] . \quad (30)$$

For tunneling of a particle through the horizon the temporal coordinates suffers a rotation in the complex plane and hence eqns. (29) and (30) shows the corresponding contribution from the imaginary part. In the classical limit  $\hbar \rightarrow 0$  so  $P_{in}$  is unity [35] and we get

$$\text{Im } t = - \text{Im} \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \quad (31)$$

and

$$P_{out} = \exp \left[ -\frac{4}{\hbar} \omega_G \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right) \text{Im} \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right] . \quad (32)$$

Following the argument in [10] we can give a thermal interpretation to the result by comparing with the Boltzmann factor ( $e^{-\frac{\omega}{T}}$ ) and eventually obtain the quantum corrected temperature of the black hole

$$T = \frac{T_H}{\left( 1 - \frac{\alpha l_p}{2(\delta x)} + \frac{\alpha^2 l_p^2}{2(\delta x)^2} - \frac{\alpha^3 l_p^3}{2(\delta x)^3} + \dots \right) (1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}})} , \quad (33)$$

where  $T_H = \frac{\hbar}{4} \left( \text{Im} \int_0^r \frac{dr}{\sqrt{f(r) g(r)}} \right)^{-1}$  is the standard Hawking temperature with specific  $f(r)$  and  $g(r)$ . For a Schwarzschild black hole  $f(r) = g(r) = (1 - \frac{2M}{r})$  and  $r_H = 2M$  and hence the standard Hawking temperature is found to be  $T_H = \frac{\hbar}{8\pi M}$ . So far we have worked in units where  $G = c = k_B = 1$ , so we have  $l_p = \sqrt{\hbar}$ . Here we will choose  $\delta x \sim 2r_H = 4M$  (a brief argument can be found in [36, 24]). If we put this in eqn. (33) and Taylor expand the expression we get the quantum corrected temperature for the black hole as

$$T = T_H \left[ 1 + \frac{\alpha l_p}{8M} - \left( \frac{\alpha^2}{32} + \beta_1 \right) \frac{l_p^2}{M^2} + \dots \dots \dots \mathcal{O} \left( \frac{l_p^3}{M^3} \text{ or higher} \right) \right] . \quad (34)$$

Already we have some existing results in the literature. With one loop back reaction effect this modification of the Hawking temperature was obtained in [15]. Similar results can also be obtained with techniques from conformal field theory [37]. There the prefactor of the quadratic term in  $\frac{l_p}{M}$  is related to the trace anomaly ( $\alpha'$ ) which depends on the number of fields with specific spin. But in addition here we have obtained a new correction term ( $\frac{\alpha l_p}{8M}$ ) which is entirely different from the existing results contributing positively to the standard Hawking temperature.

Using the law of black hole thermodynamics we get the entropy of a Schwarzschild black hole as

$$S_{bh} = \int \frac{dM}{T} . \quad (35)$$

Putting (35) in (33) we get the quantum corrected entropy as

$$S_{bh} = \frac{A}{4l_p^2} - \frac{\sqrt{\pi}\alpha}{2} \sqrt{\frac{A}{4l_p^2}} + 4\pi \left( \beta_1 + \frac{\alpha^2}{32} \right) \ln \frac{A}{4l_p^2} + 2\pi^{3/2} \alpha \beta_1 \frac{1}{\sqrt{\frac{A}{4l_p^2}}} - \frac{\pi^2 \alpha^2 \beta_1}{2} \frac{1}{\frac{A}{4l_p^2}} + \dots \dots \dots , \quad (36)$$

where  $A = 4\pi r_H^2 = 16\pi M^2$  is the area of the event horizon.  $\frac{A}{4l_p^2}$  is the standard Bekenstein-Hawking entropy. We rewrite eqn. (36) in the form of an expansion as

$$S \simeq \frac{A}{4l_p^2} - \frac{\pi^{1/2}\alpha}{2} \sqrt{\frac{A}{4l_p^2}} + 4\pi \left( \beta_1 + \frac{\alpha^2}{32} \right) \ln \frac{A}{4l_p^2} \\ + \sum_{m=\frac{1}{2}, \frac{3}{2}, \dots}^{\infty} d_m \left( \frac{A}{4l_p^2} \right)^{-m} - \sum_{n=1, 2, \dots}^{\infty} c_n \left( \frac{A}{4l_p^2} \right)^{-n} + \text{const.} \quad . \quad (37)$$

Here  $m$  denotes positive half-integers and  $n$  positive integers. If we compare this equation with (2) we can see that there are extra terms in this equation. One of the leading contribution to the entropy is from the new second term  $\sim \sqrt{\text{Area}}$ . Also we have other new correction terms proportional to  $(\text{Area})^{-m}$ . This was first pointed out in [29] and later in [38] for the same situation.

So in this Letter we study the effects of the Generalized Uncertainty Principle in black hole tunneling formalism as recently developed. We applied the Hamilton-Jacobi method for the calculation of the imaginary part of the action and the GUP is introduced through the correction to the energy of a particle due to gravity in the immediate vicinity of the horizon. Later we calculated the quantum corrected temperature for a Schwarzschild black hole and found some new correction terms as compared to the existing results in the literature. The effect of these new corrections remained in the expression of the Bekenstein-hawking entropy and the leading order correction being  $\sim \sqrt{A}$ , where  $A$  is the area of the event horizon. Our leading order correction is different and hence do not agree with the entropy bound as given by the local quantum field theory or the holographic principle [39, 40]. Although it can be shown that the holographic principle (with regard to cosmology) will remain valid as long as our universe (if flat or open) is *non-planckian*. The philosophy of interpretation of the new leading order  $\sqrt{A}$ - type correction in black-hole entropy also seems to be awaiting a new direction.

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